Lecture 24. Diagonalization

Def Let A be an $n \times n$ matrix.

(1) A diagonalization of A is an expression

$$\boxed{A = PDP^{-1}}$$

with an invertible matrix P and a diagonal matrix D.

(2) An eigenbasis for A is a basis of $\mathbb{R}^n$ whose elements are all eigenvectors of A.

Thm An an $n \times n$ matrix A has a diagonalization

$\iff$ $\mathbb{R}^n$ has an eigenbasis for A.

pf Take the linear transformation T with standard matrix A

$\mathbb{R}^n$ has an eigenbasis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ for A

(with $A\vec{v}_i = \lambda_i \vec{v}_i$ for some $\lambda_i \in \mathbb{R}$)

$\iff$ $\mathbb{R}^n$ has a basis $\mathcal{B}$ such that the $\mathcal{B}$-matrix of T is

$$A_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \text{ diagonal}$$

$\iff$ $A = PDP^{-1}$ with $D = A_{\mathcal{B}}$ being diagonal

Note For a diagonalization $A = PDP^{-1}$, the columns of P form an eigenbasis for A with the corresponding eigenvalues given by the diagonal entries of D.

Prop Let A be an $n \times n$ matrix.

(1) Eigenvectors of A that correspond to distinct eigenvalues are linearly independent.

(2) $\mathbb{R}^n$ has an eigenbasis for A

$\iff$ The geometric multiplicities of all eigenvalues add up to n

Note If $\mathbb{R}^n$ has an eigenbasis for A, it is formed by merging bases of all eigenspaces.

Prop If an $n \times n$ matrix A has n distinct eigenvalues, it has a diagonalization.

pf A has n distinct eigenvalues $\alpha_1, \alpha_2, \cdots, \alpha_n$

$\Longrightarrow$ $p_A(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\cdots(\lambda - \alpha_n)$

$\Longrightarrow$ All eigenvalues of A have algebraic multiplicity 1

$\Longrightarrow$ All eigenvalues of A have geometric multiplicity 1

$\Longrightarrow$ The geometric multiplicities of all eigenvalues add up to n

$(1 + 1 + \cdots + 1 = 1 \cdot n = n)$

$\Longrightarrow$ $\mathbb{R}^n$ has an eigenbasis for A

$\Longrightarrow$ A has a diagonalization

Note It is possible that A has a diagonalization without having n distinct eigenvalues.

Ex If possible, diagonalize each matrix by writing it as PDP^{-1} with an invertible matrix P and a diagonal matrix D.

(1) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Sol $p_A(\lambda) = \lambda^2 - (2+2)\lambda + (2\cdot 2 - 1\cdot 1) = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$

$\Longrightarrow$ A has eigenvalues $\lambda = 1, 3$

$\Longrightarrow$ A is diagonalizable (2×2 matrix with 2 distinct eigenvalues)

$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Longrightarrow \text{RREF}(A-I) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$(A-I)\vec{x} = \vec{0} \Longrightarrow x_1 + x_2 = 0 \Longrightarrow x_1 = -x_2 \underset{x_2 = t}{\Longrightarrow} \vec{x} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Longrightarrow$ $\text{Nul}(A-I)$ has a basis given by $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Longrightarrow \text{RREF}(A-3I) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$(A-3I)\vec{x} = \vec{0} \Longrightarrow x_1 - x_2 = 0 \Longrightarrow x_1 = x_2 \underset{x_2 = t}{\Longrightarrow} \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Longrightarrow$ $\text{Nul}(A-3I)$ has a basis given by $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence A has a diagonalization $A = PDP^{-1}$ with

$$\boxed{P = \begin{array}{c} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ \vec{v} \quad \vec{w} \end{array} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}$$

eigenvalues for $\vec{v}, \vec{w}$

(in order!)

(2) $B = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Sol B is triangular $\Rightarrow$ B has eigenvalues $\lambda = 1, 3$ (diagonal entries)

$\lambda = 1: \; B - I = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{RREF}(B-I) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ col 3 with no leading 1's

$\Rightarrow \dim(\text{Nul}(B-I)) = 1$

$\lambda = 3: \; B - 3I = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow \text{RREF}(B-3I) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ col 1, col 2 with no leading 1's

$\Rightarrow \dim(\text{Nul}(B-3I)) = 2$

B is diagonalizable. (sum of geometric multiplicities: $1 + 2 = 3$)

$(B-I)\vec{x} = \vec{0} \Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -2x_3 \end{cases} \underset{x_3 = t}{\Rightarrow} \vec{x} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$\Rightarrow$ Nul$(B-I)$ has a basis given by $\vec{u} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$(B-3I)\vec{x} = \vec{0} \Rightarrow x_3 = 0 \underset{\substack{x_1 = s \\ x_2 = t}}{\Rightarrow} \vec{x} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ Nul$(B-3I)$ has a basis given by $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Hence B has a diagonalization $B = PDP^{-1}$ with

$$\boxed{P = \begin{array}{c} \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \vec{u} & \vec{v} & \vec{w} \end{array} \end{array} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

eigenvalues for $\vec{u}, \vec{v}, \vec{w}$ (in order!)

(3) $C = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Sol C is triangular $\Rightarrow$ C has eigenvalues $\lambda = 1, 2$ (diagonal entries)

$\lambda = 1$: $C - I = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{RREF}(C-I) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ col 3 with no leading 1's

$\Rightarrow \dim(\text{Nul}(C-I)) = 1$

$\lambda = 2$: $C - 2I = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \text{RREF}(C-2I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ col 1 with no leading 1's

$\Rightarrow \dim(\text{Nul}(C-2I)) = 1$

C is $\boxed{\text{not diagonalizable}}$ (sum of geometric multiplicities: $1 + 1 = 2$)